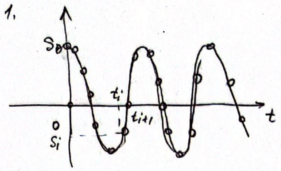
TD3

Ex. 1

1.



2. $t_{i+1} - t_i = \tau$

3. $$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + o((x - x_0)^2)$$

$$S_{i+1} = S(t_{i+1}) = S(t_i + \tau) = S(t_i) + S'(t_i)(t_i + \tau - t_i) + o(\tau)$$
$$= S_i + \tau S'(t_i) + o(\tau)$$

$$S'(t_i) \approx \frac{S_{i+1} - S_i}{\tau}$$

4. $$S_{i-1} = S(t_{i-1}) = S(t_i - \tau) = S(t_i) + S'(t_i)(\cancel{t_i} - \tau - \cancel{t_i}) + \frac{S''(t_i)}{2}\cdot(-\tau)^2 + o(\tau^2)$$
$$= S_i - \tau S'(t_i) + \frac{\tau^2}{2}S''(t_i) + o(\tau^2)$$
$$S_{i+1} = S(t_{i+1}) = S(t_i + \tau) = S(t_i) + \tau S'(t_i) + \frac{\tau^2}{2}S''(t_i) + o(\tau^2)$$
$$= S_i + \tau S'(t_i) + \frac{\tau^2}{2}S''(t_i) + o(\tau^2)$$

derivée second

$i-1 \quad i \quad i+1$

$$S_{i-1} + S_{i+1} = 2S_i + \tau^2 S''(t_i) + \underbrace{o(\tau^2)}_{= \tau^2 \varepsilon(\tau)}$$

$$S''(t_i) = \frac{S_{i+1} - 2S_i + S_{i-1}}{\tau^2} + \underbrace{o(1)}_{= \underbrace{\tau^0}_{=1} \varepsilon(\tau)}$$

$$S''(t_i) \approx \frac{S_{i+1} - 2S_i + S_{i-1}}{\tau^2}$$

derivée 3ème

$i-2 \quad i-1 \quad i \quad i+1 \quad i+2$

5. $$S_{i+2} = S_i + 2\tau S'(t_i) + \frac{\cancel{4}^{2}\tau^2}{\cancel{2}} S''(t_i) + \frac{\cancel{8}^{4}\tau^3}{\cancel{6}_{3}} S'''(t_i) + o(\tau^3)$$
$$S_{i+1} = S_i + \tau S'(t_i) + \frac{\tau^2}{2}S''(t_i) + \frac{\tau^3}{6}S'''(t_i) + o(\tau^3)$$
$$S_{i-1} = S_i - \tau S'(t_i) + \frac{\tau^2}{2}S''(t_i) - \frac{\tau^3}{6}S'''(t_i) + o(\tau^3)$$
$$S_{i-2} = S_i - 2\tau S'(t_i) + \frac{\cancel{4}^{2}\tau^2}{\cancel{2}} S''(t_i) - \frac{\cancel{8}^{4}\tau^3}{\cancel{6}_{3}} S'''(t_i) + o(\tau^3)$$

$$\begin{cases} a + b + c + d = 0 \\ 2a + b - c - 2d = 0 \\ 2a + \frac{1}{2}b + \frac{1}{2}c + 2d = 0 \end{cases}$$

$b = 2d$
$c = -2a$
$a = -d$

$a = 1$
$b = -2$
$c = 2$
$d = -1$

$$S_{i+2} \approx \cancel{S_i} + \cancel{2\tau S'} + \cancel{2\tau^2 S''} + \frac{4}{3}\tau^3 S'''$$
$$-2S_{i+1} \approx \cancel{-2S_i} - \cancel{2\tau S'} - \cancel{\tau^2 S''} - \frac{1}{3}\tau^3 S'''$$
$$+ \quad 2S_{i-1} \approx \cancel{2S_i} - \cancel{2\tau S'} + \cancel{\tau^2 S''} - \frac{1}{3}\tau^3 S'''$$
$$-S_{i-2} \approx -\cancel{S_i} + \cancel{2\tau S'} - \cancel{2\tau^2 S''} + \frac{4}{3}\tau^3 S'''$$

$$S_{i+2} - 2S_{i+1} + 2S_{i-1} - S_{i-2} = 2\tau^3 S'''(t_i)$$

$$S'''(t_i) \approx \frac{S_{i+2} - 2S_{i+1} + 2S_{i-1} - S_{i-2}}{2\tau^3}$$

Ex. 2

1. $F = -\alpha \vec{v}$ frottement faible $\alpha \approx 0$. On doit retrouver $\dot{z}(t) = -gt$ pour $\alpha \to 0$.

$e^x = 1 + x + \frac{x^2}{2} + o(x^2) \quad x \approx 0$

$$-\frac{mg}{\alpha}\left(1 - e^{-\alpha t/m}\right) = -\frac{mg}{\alpha}\left(1 - \left(1 - \frac{\alpha t}{m}\right) + o\left(\frac{\alpha t}{m}\right)\right) = -\frac{mg}{\alpha}\left(\cancel{1} - \cancel{1} + \frac{\alpha t}{m} + o\left(\frac{\alpha t}{m}\right)\right)$$

$x = -\frac{\alpha t}{m} \approx 0$

$$= -gt + \underbrace{\frac{mg}{\cancel{\alpha}}\cdot\frac{\cancel{\alpha} t}{\cancel{m}}\varepsilon(\alpha)}_{\xrightarrow[\alpha \to 0]{} 0}$$

$$\xrightarrow[\alpha \to 0]{} -gt$$

Donc, $\lim_{\alpha \to 0} \dot{z}(t) = -gt$

2. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + o(x^5)$

$$-\sqrt{\frac{mg}{\alpha}}\tan\left(\sqrt{\frac{\alpha g}{m}}t\right) = -\sqrt{\frac{mg}{\alpha}}\left(\sqrt{\frac{\alpha g}{m}}t + o\left(\sqrt{\frac{\alpha g}{m}}t\right)\right) = -\sqrt{\frac{m g}{\cancel{\alpha}}\cdot\frac{\cancel{\alpha} g}{\cancel{m}}}\,t$$

$\sqrt{g^2}$

$$-\sqrt{\frac{\cancel{m}g}{\cancel{\alpha}}}\cdot\sqrt{\frac{\cancel{\alpha}g}{\cancel{m}}}\,t\,\varepsilon(\sqrt{\alpha}) = -gt - gt\,\underbrace{\varepsilon(\sqrt{\alpha})}_{\xrightarrow[\alpha \to 0]{} 0}$$

. Donc, de même $\lim_{\alpha \to 0} \dot{z}(t) = -g(t)$

Utilité : Verification de la modélisation du terme de frottement !

Ex. 3

$\beta = \frac{v}{c} \qquad \gamma = \frac{1}{\sqrt{1 - \beta^2}}$

1. $E = \gamma mc^2 \qquad v \ll c \qquad \beta \ll 1$

$$E = \frac{mc^2}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = mc^2 \frac{1}{\sqrt{1 - \beta^2}}$$

$$= mc^2 \frac{1}{1 \underbrace{- \frac{\beta^2}{2} + o(\beta^2)}_{= u}} =$$

$$= mc^2\left(1 + \frac{\beta^2}{2} + o(\beta^2)\right) = mc^2 + mc^2\frac{v^2}{2c^2} + o\left(\left(\frac{v}{c}\right)^2\right)$$

$$= \underbrace{mc^2}_{\text{de masse}} + \underbrace{\frac{mv^2}{2}}_{\text{cinetique}} + o\left(\left(\frac{v}{c}\right)^2\right)$$

energie de masse, — cinetique

Taylor : $f(x) = \sqrt{1 - x} \qquad x \approx 0.$

$f(0) = 1$

$f'(x) = \frac{-1}{2\sqrt{1 - x}} = -\frac{1}{2}(1 - x)^{-1/2}$

$f''(x) = -\frac{1}{2}\cdot\left(-\frac{1}{2}\right)(1 - x)^{-3/2} \times (-1) \qquad f''(0) = -\frac{1}{4}$

$f(x) = 1 - \frac{1}{2}x - \frac{1}{4}\cdot\frac{1}{2!}x^2 + o(x^2) = 1 - \frac{x}{2} - \frac{x^2}{8} + o(x^2)$

$\frac{1}{1 - u} = 1 + u + u^2 + o(u^2)$

2. $f = f_0\sqrt{\frac{1 - \beta}{1 + \beta}} \qquad \beta \approx 0$

$$f_0\sqrt{(1 - \beta)\cdot\underbrace{\frac{1}{1 + \beta}}_{= \frac{1}{1 - (-\beta)}}} = f_0\sqrt{(1 - \beta)(1 - \beta + o(\beta))} = f_0\sqrt{1 - 2\beta + o(\beta)} = f_0\left(1 - \frac{1}{2}\cdot 2\beta\right) + o(\beta)$$

$$= f_0\left(1 - \frac{v}{c}\right) + o(\beta)$$

1 Calcul mental

$a = 0.99^3 = (1-0.01)^3 \approx 1 - 3\times 0{,}01 = 0{,}97$

$f(x) = (1-x)^3 \quad x_0 = 0 \qquad (1-x)^3 = (1-0)^3 - \underbrace{3(1-0)^2}_{=3}\underbrace{(x-x_0)}_{=x} + o((x-x_0)^2)$

$f'(x) = 3(1-x)^2 \times (-1) = -3(1-x)^2$

un peu moins que $\frac{1}{10}$.

$b = \dfrac{1}{10{,}1} = \dfrac{1}{10+0{,}1} = \dfrac{1}{10}\cdot\dfrac{1}{1-(-0{,}01)} = \dfrac{1}{10}\cdot\left(1 - 0{,}01 + o(0{,}01)\right)$

$\approx 0{,}099$

$\approx \frac{2}{10}$

$c = \dfrac{1}{10{,}1} + \dfrac{1}{9{,}9} = \dfrac{1}{10+0{,}1} + \dfrac{1}{10-0{,}1} = \dfrac{1}{10}\cdot\dfrac{1}{1+0{,}01} + \dfrac{1}{10}\cdot\dfrac{1}{1-0{,}01}$

$= \dfrac{1}{10}\left(\dfrac{1}{1-(-0{,}01)} + \dfrac{1}{1-0{,}01}\right) = \dfrac{1}{10}\Big(1 - \cancel{0{,}01} + (0{,}01)^2 + o((0{,}01)^2) + 1 + \cancel{0{,}01} + (0{,}01)^2 + o((0{,}01)^2)\Big)$

$= \dfrac{1}{10}(2 + 0{,}0002) = 0{,}20002$

Ex5 1. $f(x) = \ln(x^2+2x+2)$ en $x_0 = 0$.

i). Taylor (calcul directe)

$f(x) = \ln(x^2+2x+2)$; $f(0) = \ln(2)$

$f'(x) = \dfrac{1}{x^2+2x+2}\cdot 2x+2 = \dfrac{2(x+1)}{x^2+2x+2}$; $f'(0) = 1$

$4(x+1)^2 = 4(x^2+2x+1)$

$f''(x) = \dfrac{2(x^2+2x+2) - (2x+2)2(x+1)}{(x^2+2x+2)^2} = \dfrac{2x^2+4x+\cancel{4} - 4x^2 - 8x - \cancel{4}}{(x^2+2x+2)^2} = \dfrac{-2x^2-4x}{(x^2+2x+2)^2} = \dfrac{-2(x^2+2x)}{(x^2+2x+2)^2}$

$f''(0) = 0.$

$f'''(x) = \dfrac{(-4x-4)(x^2+2x+2)^2 - 2(x^2+2x+2)(2x+2)(-2x^2-4x)}{(x^2+2x+2)^4}$

$f'''(0) = \dfrac{-4\cdot 2^2}{2^4} = \dfrac{-16}{16} = -1$

$f(x) = \ln 2 + x + 0\cdot\dfrac{x^2}{2} - 1\,\dfrac{x^3}{3!} + o(x^3) = \ln 2 + x - \dfrac{x^3}{6} + o(x^3)$

ii) Utiliser $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + o(x^4)$

$\ln(x^2+2x+2) = \ln\left(2\left(\dfrac{x^2}{2}+x+1\right)\right) = \ln(2) + \ln\left(1+x+\dfrac{x^2}{2}\right)$

$= \ln(2) + x + \dfrac{x^2}{2} \underbrace{- \dfrac{1}{2}\left(x+\dfrac{x^2}{2}\right)^2 + \dfrac{1}{3}\left(x+\dfrac{x^2}{2}\right)^3}_{\text{troncature}} + o(x^3)$

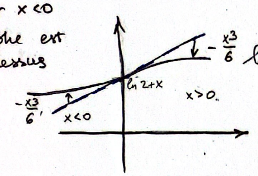
$= \ln(2) + x + \dfrac{x^2}{2} - \dfrac{1}{2}\left(x^2 + \cancel{2}\cdot x\cdot\dfrac{x^2}{\cancel{2}}\right) + \dfrac{1}{3}(x^3) + o(x^3)$

l'ordre suivant

$= \ln(2) + x + \cancel{\dfrac{x^2}{2}} - \cancel{\dfrac{x^2}{2}} - \dfrac{x^3}{2} + \dfrac{x^3}{3} + o(x^3) = \underbrace{\ln 2 + x}_{\text{ordre 1}} - \overline{\dfrac{x^3}{6}} + o(x^3)$

ordre 1 fonction affine correspondante à la tangente en $x = 0$.

Pour $x<0$ le graphe est au-dessus



Pour $x>0$ le graphe est au-dessous de sa tangente

Ex.6 $f(x) = \sqrt{x^2+x+1}$ $\qquad h = \frac{1}{x} \qquad F(h) = f(x)$

1. $F(h) = F\left(\frac{1}{x}\right) = f(x) = f\left(\frac{1}{h}\right) = \sqrt{\dfrac{1}{h^2} + \dfrac{1}{h} + 1}$

2. $h>0$. $(h\to 0 \Leftrightarrow x \to +\infty)$ $\qquad F(h) = F(0) + F'(0)\,h + \dfrac{F''(0)}{2}h^2$ $\qquad$ sinon utiliser $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + o(x^2)$

$\sqrt{\dfrac{1}{h^2}(1+h+h^2)} = \dfrac{1}{h}\sqrt{1+\underbrace{h+h^2}_{x}} = \dfrac{1}{h}\cdot\left(1 + \dfrac{h+h^2}{2} - \dfrac{1}{8}(h+h^2)^2 + o(h^2)\right)$

$= \dfrac{1}{h}\left(1 + \dfrac{h}{2} + \dfrac{h^2}{2} - \dfrac{1}{8}h^2 + o(h^2)\right) = \dfrac{1}{h}\left(1 + \dfrac{h}{2} + \dfrac{3h^2}{8} + o(h^2)\right) = \dfrac{1}{h} + \dfrac{1}{2} + \dfrac{3h}{8} + o(h)$

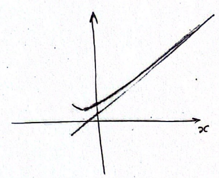
$o(h) \underset{h\to 0}{\to} 0$

$o\left(\frac{1}{x}\right) \underset{x\to+\infty}{\to} 0$

Alors $f(x) = x + \dfrac{1}{2} + \dfrac{3}{8x} + o\left(\dfrac{1}{x}\right)$

Alors $f(x) - \left(x + \dfrac{1}{2}\right) = \left(\dfrac{3}{8x} + o\left(\dfrac{1}{x}\right)\right) \underset{x\to+\infty}{\to} 0$

$y(x) = x + \frac{1}{2}$ l'asymptote au f en $+\infty$



$\dfrac{3h}{8} + \underbrace{o(h)}_{\substack{\to 0 \\ h\to 0}} \qquad h>0$

$h\left(\dfrac{3}{8} + \varepsilon(h)\right) \underset{h\to 0}{\to} 0$ $\qquad$ f est donc au-dessus de son asymptote $y(x) = x + \frac{1}{2}$

3. $h < 0$.

$$F(h) = \sqrt{\frac{1}{h^2}}\sqrt{1+h+h^2} \underset{h<0}{=} \left(-\frac{1}{h}\right)\left(1 + \frac{h}{2} + \frac{3h^2}{8} + o(h^2)\right)$$

$$= -\frac{1}{h} - \frac{1}{2} - \frac{3h}{8} + o(h)$$

donc $f(x) = -x - \frac{1}{2} - \frac{3}{8x} + o\left(\frac{1}{x}\right)$

l'asymptote en $-\infty$: $y(x) = -x - \frac{1}{2}$ et $-\frac{3}{8x} + o\left(\frac{1}{x}\right) > 0$ pour $x \to -\infty$

donc f est au-dessus de son asymptote.